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ALMOST GLOBAL EXISTENCE FOR SOME SEMILINEAR WAVE EQUATIONS WITH ALMOST CRITICAL REGULARITY

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Abstract. For any subcritical index of regularity $s > 3/2$, we prove the almost global well posedness for the 2-dimensional semilinear wave equation with the cubic nonlinearity in the derivatives, when the initial data are small in the Sobolev space $H^s \times H^{s-1}$ with certain angular regularity. The lifespan is known to be sharp in general. The main new ingredient in the proof is an endpoint version of the generalized Strichartz estimates in the space $L^2_t L^\infty_{|x|} L^2_\theta([0, T] \times \mathbb{R}^2)$. In the last section, we also consider the general semilinear wave equations with the spatial dimension $n \geq 2$ and the order of nonlinearity $p \geq 3$.

1. Introduction

The purpose of this paper is, for the 2-dimensional semilinear wave equation with the cubic nonlinearity in the derivatives, to prove the almost global well posedness with low regularity and sharp lifespan. The main new ingredient in the proof will be an endpoint version of the generalized Strichartz estimates in the space $L^2_t L^\infty_{|x|} L^2_\theta([0, T] \times \mathbb{R}^2)$. As complement, we will also consider the general semilinear wave equations with the spatial dimension $n \geq 2$ and the order of nonlinearity $p \geq 3$ in the last section.

Let $\Box \equiv \partial_t^2 - \Delta$, $\partial = (\partial_t, \partial_x)$ and P_α be polynomials for $\alpha \in \mathbb{N}^3$, we consider the following Cauchy problem

$$\Box u = \sum_{|\alpha|=3} P_\alpha(u)(\partial u)^\alpha \tag{1.1}$$

on $[0, T] \times \mathbb{R}^2$, together with the initial data at time $t = 0$

$$u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x)\ . \tag{1.2}$$

In the case of classical C_0^∞ initial data with size of order ϵ, the almost global existence with

$$T_\epsilon \geq \exp(c\epsilon^{-2}) \tag{1.3}$$

(for some small constant $c > 0$) can be proved by the standard energy methods, see e.g. Sogge [10]. Moreover, the lifespan T_ϵ is also sharp for the problem with nonlinearity $|\partial_t u|^3$ (Zhou [17]).

Our object here is to prove the corresponding result with low regularity. Note that the equation (1.1) with P_α be constants C_α is invariant under the scaling

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transformation $u(t,x) \to \lambda^{-\frac{1}{2}} u(\lambda t, \lambda x)$. This scaling preserves the critical Sobolev space $\dot{H}^{s_c}$ with exponent

$$s_c = \frac{3}{2}\ , \tag{1.4}$$

which is then, heuristically, a lower bound for the range of admissible s such that the problem (1.1)-(1.2) is well-posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$. (See e.g. Theorem 2 in [3] for the ill posed result with $s < s_c$ and nonlinearity $(\partial_t u)^3$.)

The local well posedness for the problem of this type with low regularity has been extensively studied (see Ponce-Sideris [9], Tataru [16] and the authors [1]). For this problem, besides scaling, there is one more mechanism due to Lorentz invariance such that the problem is not well posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$ with $s = s_c + \epsilon$ for arbitrary small $\epsilon \ll 1$ (see e.g. Lindblad [6]). Instead, the local well posedness is true for $s > \frac{7}{4}$.

To state our main result, we need to introduce the Sobolev space with angular regularity $b > 0$,

$$f \in H^{s,b}_\theta \Leftrightarrow f \in H^s, \text{ and } (1-\partial^2_\theta)^{b/2} f \in H^s \tag{1.5}$$

where the (r,θ) is the polar coordinates. Now we are ready to state our main result.

Theorem 1.1. *Let $n = 2$, $s > s_c = 3/2$ and $b > 1/2$. Then there exist two small positive constant ϵ_0 and c, such that the problem* (1.1)-(1.2) *admits an unique almost global solution $(u, \partial_t u) \in C_{T_\epsilon}(H^{s,b}_\theta \times H^{s-1,b}_\theta)$ with $\partial_{t,x} u \in L^2_{T_\epsilon} L^\infty_{|x|} H^b_\theta$ on $[0, T_\epsilon] \times \mathbb{R}^2$ with $T_\epsilon = \exp(c\epsilon^{-2})$, whenever $(u_0, u_1) \in H^{s,b}_\theta \times H^{s-1,b}_\theta$ with norm bounded by $\epsilon \le \epsilon_0$.*

Remark 1. Here we note that, by adding some angular regularity, the Sobolev regularity required to ensure almost global existence is only $s > s_c$, which is $1/4$ less than the usual requirement of $s > 7/4$.

To prove Theorem 1.1 we shall require certain Strichartz estimates, which involve the angular mixed-norm spaces

$$\|u\|_{L^q_t L^\infty_{|x|} L^2_\theta(\mathbb{R}^2)} = \left(\int_{\mathbb{R}} \operatorname{esssup}_{\rho>0} \left(\int_0^{2\pi} |f(\rho(\cos\theta, \sin\theta))|^2\, d\theta \right)^{q/2} dt \right)^{1/q}.$$

Theorem 1.2. *Let $P = \sqrt{-\Delta}$ in $\mathbb{R}^2$. Then for any $\gamma > \frac{1}{2}$, there exists a constant C_γ such that*

$$\left\| e^{-itP} f \right\|_{L^2_t L^\infty_{|x|} L^2_\theta([0,T]\times\mathbb{R}^2)} \le C_\gamma \left(\ln(2+T)\right)^{\frac{1}{2}} \|f\|_{H^\gamma(\mathbb{R}^2)}. \tag{1.6}$$

Moreover, if $2 < q < \infty$, then

$$\left\| e^{-itP} f \right\|_{L^q_t L^\infty_{|x|} L^2_\theta(\mathbb{R}\times\mathbb{R}^2)} \le C_q \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)}, \quad \gamma = 1 - 1/q. \tag{1.7}$$

The estimate (1.6) can be viewed as the endpoint estimate of the estimates (1.7), which were proved recently in Smith, Sogge and the second author [11]. We mention that the related estimates for (1.7) where L^2_θ is replaced by L^r_θ (with norms of different regularity in the right) were proved by Sterbenz [14] for $n \ge 4$ and the authors [4] for the general case $n \ge 2$. In the radial case, the estimates (1.7) and the higher dimensional version were proved by the authors in [2] (with previous works in Sogge [10] for $n = 3$, Sterbenz [14] for $n \ge 3$).

Remark 2. When $4 < q < \infty$, the Strichartz estimates (1.7) is weaker than the standard Strichartz estimates (see Theorem 3 of [2])

$$\left\| e^{-itP} f \right\|_{L^q_t L^\infty_x(\mathbb{R}\times\mathbb{R}^2)} \le C_q \|f\|_{\dot H^\gamma(\mathbb{R}^2)}, \quad \gamma = 1 - 1/q,\ 4 < q < \infty. \tag{1.8}$$

By interpolating (1.7) with (1.8), we can also improve L^2_θ to L^p_θ in (1.7).

Remark 3. Note that we have also the trivial energy estimate

$$\left\| e^{-itP} f \right\|_{L^\infty_t L^2_{|x|} L^2_\theta} \le C\|f\|_{L^2}, \tag{1.9}$$

since e^{-itP} is an unitary operator on L^2. By interpolation, we can also get more general Strichartz type estimates involving $L^q_t L^r_{|x|} L^2_\theta$ norm, where

$$\|u\|_{L^q_t L^r_{|x|} L^2_\theta(\mathbb{R}\times\mathbb{R}^2)} = \left(\int_{\mathbb{R}} \left(\int_0^\infty \left(\int_0^{2\pi} |f(\rho(\cos\theta, \sin\theta))|^2 \, d\theta \right)^{r/2} \rho d\rho \right)^{q/r} dt \right)^{1/q}.$$

This paper is organized as follows. In the next section, we give a proof of Theorem 1.2, inspired by the arguments of Smith, Sogge and the second author [11] and Sterbenz [14]. In Section 3, we shall prove the almost global wellposedness with small data for the problem (1.1)-(1.2). To deal with the general nonconstant functions P_α, we will need to obtain improved bound for the solution u, which is achieved in Lemma 3.2. In the final section, for the semilinear wave equations with general dimension and general nonlinearity, we shall exploit further the applications of the classical Strichartz estimates and their angular improvement in Theorem 1.2, as an appendix to the wellposed result for the 2-dimensional cubic semilinear wave equation.

2. Strichartz estimates

In this section, we prove Theorem 1.2, including the critical $L^2_t L^\infty_{|x|} L^2_\theta$ Strichartz estimates for the wave equation when $n = 2$. We split the proof into three steps. Although these steps are essentially the same except the last step as in [11], we write out the complete proof for the sake of completeness.

2.1. **Frequency Localization.** At first, we want to reduce the inequalities to the frequency localized counterparts.

It is easy to see that the frequency localized estimates for Theorem 1.2 are as follows

$$\|e^{-itP} f\|_{L^q_t L^\infty_{|x|} L^2_\theta(\mathbb{R}\times\mathbb{R}^2)} \le C_q \|f\|_{L^2(\mathbb{R}^2)}, \text{ if } q > 2, \text{ and } \hat f(\xi) = 0,\ |\xi| \notin [1/2, 1] \tag{2.1}$$

and

$$\|e^{-itP} f\|_{L^2_t L^\infty_{|x|} L^2_\theta([0,T]\times\mathbb{R}^2)} \le C(\ln(2+T))^{1/2} \|f\|_{L^2(\mathbb{R}^2)}, \text{ if } \hat f(\xi) = 0,\ |\xi| \notin [1/2, 1] \tag{2.2}$$

if $P = \sqrt{-\Delta}$.

By scaling and Littlewood-Paley theory, we see that (1.7) and (2.1) are equivalent. To deduce (1.6) from (2.2), we will need to verify the following estimate for any $\delta > 0$

$$\sum_{j\in\mathbb{Z}} 2^{j/2}(1+2^j)^{-1/2-\delta} (\ln(2+2^j T))^{1/2} \le C_\delta (\ln(2+T))^{1/2}. \tag{2.3}$$

In fact, if $T \geq e$, we deal with the following two different cases.
i) $2^j \geq 1$;

$$\begin{aligned}\sum_{j\geq 0} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\leq \sum_{j\geq 0} 2^{-j\delta}(\ln(2+2^jT))^{1/2} \\ &\leq C\sum_{j\geq 0} 2^{-j\delta}(j\ln 2+\ln T)^{1/2} \\ &\leq C\sum_{j\geq 0} 2^{-j\delta}(j\ln 2+1)^{1/2}(\ln T)^{1/2} \\ &\leq C_\delta(\ln T)^{1/2}.\end{aligned}$$

ii) $2^j \leq 1$;

$$\begin{aligned}\sum_{j<0} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\leq \sum_{j<0} 2^{j/2}(\ln(2+2^jT))^{1/2} \\ &\leq \sum_{j<0} 2^{j/2}(\ln(2+T))^{1/2} \\ &\leq C(\ln(2+T))^{1/2}.\end{aligned}$$

Else, if $T \leq e$, we also deal with two different cases.
i) $2^j \geq T^{-1}$;

$$\begin{aligned}\sum_{2^jT\geq 1} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\leq \sum_{2^jT\geq 1} 2^{-j\delta}(\ln(2+2^jT))^{1/2} \\ &\leq T^\delta \sum_{2^jT\geq 1} (2^jT)^{-\delta}(\ln(2+2^jT))^{1/2} \\ &\leq C_\delta T^\delta \leq \tilde{C}_\delta.\end{aligned}$$

ii) $1 \leq \lambda = 2^j \leq T^{-1}$;

$$\sum_{2^jT<1} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} \leq C\sum_j 2^{j/2}(1+2^j)^{-1/2-\delta} \leq C_\delta.$$

2.2. **Further reduction.** Let us turn to the proof of (2.1) and (2.2). Due to the support assumptions for $\hat{f}$ we have that

$$\|f\|^2_{L^2(\mathbb{R}^2)} \approx \int_0^\infty \int_0^{2\pi} |\hat{f}(\rho(\cos\omega,\sin\omega))|^2 d\omega d\rho. \tag{2.4}$$

If we expand the angular part of $\hat{f}$ using Fourier series we find that if $\xi = \rho(\cos\omega,\sin\omega)$ then there are Fourier coefficients $c_k(\rho)$ which vanish when $\rho \notin [1/2, 1]$ so that

$$\hat{f}(\xi) = \sum_k c_k(\rho)e^{ik\omega},$$

and so, by (2.4) and Plancherel's theorem for $\mathbb{S}^1$ and $\mathbb{R}$ we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} \approx \sum_k \int_{\mathbb{R}} |c_k(\rho)|^2\, d\rho \approx \sum_k \int_{\mathbb{R}} |\hat{c}_k(s)|^2\, ds, \tag{2.5}$$

if $\hat{c}_k$ denotes the one-dimensional Fourier transform of $c_k(\rho)$. Recall that (see Stein and Weiss [12] p. 137)

$$f(r(\cos\theta,\sin\theta)) = \frac{1}{2\pi}\sum_k \Big(i^k \int_0^\infty J_k(r\rho)c_k(\rho)\rho\, d\rho\Big) e^{ik\theta}, \tag{2.6}$$

if J_k is the k-th Bessel function, i.e.,

$$J_k(y) = \frac{(-i)^k}{2\pi}\int_0^{2\pi} e^{iy\cos\theta - ik\theta}\, d\theta. \tag{2.7}$$

Because of (2.6) and the support properties of the c_k, we find that if we fix $\beta \in C_0^\infty(\mathbb{R})$ satisfying $\beta(\tau) = 1$ for $1/2 \le \tau \le 1$ but $\beta(\tau) = 0$ if $\tau \notin [1/4, 2]$ then if we set $\alpha = \rho\beta(\rho) \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
&(e^{-itP}f)(r(\cos\theta,\sin\theta))\\
&\quad = \frac{1}{2\pi}\sum_k \Big(i^k \int_0^\infty J_k(r\rho)e^{-it\rho}c_k(\rho)\beta(\rho)\rho\, d\rho\Big) e^{ik\theta}\\
&\quad = \frac{1}{(2\pi)^2}\sum_k \Big(i^k \int_0^\infty \int_{-\infty}^\infty J_k(r\rho)e^{i\rho(s-t)}\hat{c}_k(s)\alpha(\rho)\, ds d\rho\Big) e^{ik\theta}\\
&\quad = \frac{1}{(2\pi)^3}\sum_k \Big(\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} e^{i\rho r\cos\vartheta}e^{-ik\vartheta}e^{i\rho(s-t)}\hat{c}_k(s)\alpha(\rho)\, d\vartheta ds d\rho\Big) e^{ik\theta}\\
&\quad = \frac{1}{(2\pi)^3}\sum_k \Big(\int_{-\infty}^\infty \int_0^{2\pi} e^{-ik\vartheta}\hat{\alpha}\big((t-s) - r\cos\vartheta\,\big)\, \hat{c}_k(s)\, d\vartheta ds\Big) e^{ik\theta}\\
&\quad = \frac{1}{(2\pi)^3}\sum_k \Big(\int_{-\infty}^\infty \hat{c}_k(s)\psi_k(t-s,r) ds\Big) e^{ik\theta},
\end{aligned}$$

where we set

$$\psi_k(m,r) = \int_0^{2\pi} e^{-ik\theta}\hat{\alpha}\big(\, m - r\cos\theta\,\big)\, d\theta. \tag{2.8}$$

As a result, we have that for any $r \ge 0$

$$\int_0^{2\pi} \Big| (e^{-itP}f)(r(\cos\theta,\sin\theta)) \Big|^2 d\theta = \frac{1}{(2\pi)^5}\sum_k \Big| \int_{-\infty}^\infty \hat{c}_k(s)\psi_k(t-s,r)\, ds \Big|^2. \tag{2.9}$$

Now we claim that we have the estimate

$$\|\psi_k(m,r)\, \langle m\rangle^{\frac{1}{2}}\, \|_{L^2_m} \le C\,, \tag{2.10}$$

where $\langle m\rangle = \sqrt{1+m^2}$ and C is independent of $k \in \mathbb{Z}$ and $r \ge 0$. If this is true, then

$$\begin{aligned}
\|(e^{-itP}f)(r,\theta)\|_{L^2_\theta} &\le C\|\hat{c}_k(s)\psi_k(t-s,r)\|_{l^2_k L^1_s}\\
&\le C\|\hat{c}_k(s)\, \langle t-s\rangle^{-1/2}\, \|_{l^2_k L^2_s}\|\, \langle t-s\rangle^{\frac{1}{2}}\, \psi_k(t-s,r)\|_{L^2_s}\\
&\le C\|\hat{c}_k(s)\, \langle t-s\rangle^{-1/2}\, \|_{l^2_k L^2_s},
\end{aligned}$$

and we can immediately get the required estimates (2.1) and (2.2), if we note that

$$\langle t-s\rangle^{-1/2} \in L^q \text{ if } q > 2, \text{ and } \|\, \langle t-s\rangle^{-1/2}\, \|_{L^2_{t\in[0,T]}} \le C(\ln(2+T))^{1/2}.$$

2.3. **The estimate for $\psi_k(m,r)$.** Now we present the proof of the key estimate (2.10) for $\psi_k(m,r)$, to conclude the proof of the Strichartz estimate in Theorem 1.2.

We begin with the proof of the following pointwise estimates (which is precisely Lemma 2.1 of [11]).

Lemma 2.1. *Let $\alpha \in \mathcal{S}(\mathbb{R})$ and $N \in \mathbb{N}$ be fixed. Then there is a uniform constant C_N, which is independent of $m \in \mathbb{R}$ and $r \geq 0$ so that the following inequalities hold. First,*

$$\int_0^{2\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C_N \langle m\rangle^{-N}, \quad \text{if } \ 0 \leq r \leq 1, \ \text{or} \ \ |m| \geq 2r. \tag{2.11}$$

If $r > 1$ and $|m| \leq 2r$ then

$$\int_0^{2\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C\Big(r^{-1} + r^{-1/2}\langle\, r - |m|\,\rangle^{-1/2}\Big). \tag{2.12}$$

Consequently, for any $\delta > 0$, we have the weaker estimate for (2.10)

$$\|\psi_k(m,r)\,\langle m\rangle^{\frac{1}{2}-\delta}\,\|_{L^2_m} \leq C_\delta\,, \tag{2.13}$$

with the constant C_δ independent of $r > 0$.

Proof. We first realize that (2.11) is trivial since $\hat{\alpha} \in \mathcal{S}$. To prove (2.12), it suffices to show that

$$\int_0^{\pi/4} |\hat{\alpha}(m - r\cos\theta)|\, d\theta + \int_{\pi-\pi/4}^{\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq Cr^{-1/2}\langle\, r - |m|\,\rangle^{-1/2}, \tag{2.14}$$

and also

$$\int_{\pi/4}^{\pi-\pi/4} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq Cr^{-1}. \tag{2.15}$$

In order to prove (2.14), it suffices to prove that the first integral is controlled by the right side. For if we apply this estimate to the function $\hat{\alpha}(-s)$, we then see that the second integral satisfies the same bounds. We can estimate the first integral if we make the substitution $u = 1 - \cos\theta$, in which case, we see that it equals

$$\begin{aligned}
\int_0^{1-1/\sqrt{2}} |\hat{\alpha}((m-r) + ru)|\, \frac{du}{\sqrt{2u - u^2}} &\leq \int_0^{1-1/\sqrt{2}} |\hat{\alpha}((m-r) + ru)|\, \frac{du}{\sqrt{u}} \\
&\leq Cr^{-1/2} \int_0^{\infty} |\hat{\alpha}((m-r) + u)|\, \frac{du}{\sqrt{u}} \\
&\leq C' r^{-1/2}\,\langle\, r - m\,\rangle^{-1/2} \\
&\leq C' r^{-1/2}\,\langle\, r - |m|\,\rangle^{-1/2},
\end{aligned}$$

as desired, which completes the proof of (2.14).

To prove (2.15) we just make the change of variables $u = r\cos\theta$ and note that $|du/d\theta| \approx r$ on the region of integration, which leads to the inequality as $\hat{\alpha} \in \mathcal{S}$.

Finally, we check that inequalities (2.11) and (2.12) imply (2.13). If $r \le 1$, it is trivial. Else, if $r \ge 1$, we can prove (2.13) as follows

$$\begin{aligned}
\|\psi_k(m,r)\langle m\rangle^{\frac{1}{2}-\delta}\|_{L^2_m}^2 &\le C + C\int_{|m|\le 2r} r^{-2}\langle m\rangle^{1-2\delta}\,dm\\
&\quad + C\int_{|m|\le 2r} r^{-1}\langle m\rangle^{1-2\delta}\langle r-|m|\rangle^{-1}\,dm\\
&\le C + Cr^{-2}\langle r\rangle^{2-2\delta}\\
&\quad + C\int_{r/2\le m\le 2r} r^{-1}\langle m\rangle^{1-2\delta}\langle r-m\rangle^{-1}\,dm\\
&\quad + C\int_{0\le m\le r/2} r^{-1}\langle m\rangle^{1-2\delta}\langle r-m\rangle^{-1}\,dm\\
&\le C + C\int_{r/2\le m\le 2r} r^{-2\delta}\langle r-m\rangle^{-1}\,dm\\
&\quad + C\int_{0\le m\le r/2} r^{-2}\langle m\rangle^{1-2\delta}\,dm\\
&\le C + Cr^{-2\delta}\ln(2+r) \le C_\delta \text{ (if } \delta>0).
\end{aligned}$$

∎

Here, we remark that the reason we need to introduce a parameter $\delta > 0$ is due to the estimate (2.14) (the bound r^{-1} will be enough for us to get the estimate with $\delta = 0$).

To prove the stronger estimate (2.10), we need to consider the effect of oscillated factor $e^{-ik\theta}$ in the definition of $\psi_k(m,r)$, and the support property of the function α.

To begin, we give some more reductions. At first, without loss of generality, we can assume $m \ge 0$. In this case, we need only to give the estimate for $\theta \in [0, \frac{3\pi}{4}]$ and $\theta \in [\frac{3\pi}{4}, \pi]$. For the case $\theta \in [\frac{3\pi}{4}, \pi]$, since $m - r\cos\theta \simeq m + r$ and $\hat\alpha \in \mathcal{S}$, the estimate is admissible for our purpose. So we need only to give a refined estimate for the integral of the type

$$I_k(m,r) = \int_0^{3\pi/4} e^{-ik\theta}\hat\alpha\big(m - r\cos\theta\big)\,d\theta \tag{2.16}$$

when $m \le 2r$, $r > 1$. Moreover, we observe from (2.14) and (2.15) that

$$|\psi_k(m,r)|,\ |I_k(m,r)| \le Cr^{-1},\ \text{if } |m| \le r/2,$$

which are also admissible estimates. This means that we need only to consider the case $r/2 < m < 2r$ with $r > 1$. Now we are ready to give the second estimate about $\psi_k(m,r)$ (which is resemble to Proposition 4.1 of [14]).

Lemma 2.2. *Let $\alpha \in \mathcal{S}$ with support in $[1/4, 2]$, $r > 1$ and $r/2 < m < 2r$. If $r < m+1$, then for any $N \ge 0$, we have*

$$|I_k(m,r)| \le C_N r^{-1/2}\langle r-m\rangle^{-N}, \tag{2.17}$$

Else if $r > m+1$ and set $d = \sqrt{r^2 - m^2}$, we have

$$|I_k(m,r)| \le Cr^{-1/2}\langle r-m\rangle^{-1/2}\Big(\langle r-m\rangle^{-1} + \min(k/d, d/k)\Big). \tag{2.18}$$

Here, when $k = 0$, the estimate is understood to be $|I_0(m,r)| \le Cr^{-1/2}\langle r-m\rangle^{-3/2}$.

Before giving the proof of Lemma 2.2, we give the proof of (2.10). By Lemma 2.1, Lemma 2.2 and the discussion before Lemma 2.2, we know that

$$|\psi_k(m,r)| \le C \begin{cases} \langle m\rangle^{-N} & |m|\ge 2r \text{ or } r\le 1 \\ r^{-1} & |m|\le r/2 \text{ and } r>1 \\ \langle r+|m|\rangle^{-N} + r^{-1/2}\langle |m|-r\rangle^{-N} & r<|m|+1, r/2\le |m|\le 2r \text{ and } r>1 \\ \langle r+|m|\rangle^{-N} + r^{-1/2}\langle |m|-r\rangle^{-3/2} & \\ \quad +r^{-1/2}\langle |m|-r\rangle^{-1/2}\min(k/d,d/k) & r\ge |m|+1, r/2\le |m|\le 2r \text{ and } r>1 \end{cases}$$

Then a simple calculation will give us the key estimate (2.10).

In fact, the case when $r\le 1$ is trivial. So we need only to consider the case with $r>1$, in which case, we write the integral into the sums as follows

$$\begin{aligned} & \int_{\mathbb{R}} |\psi_k(m,r)|^2 \langle m\rangle\, dm \\ = & \left(\int_{|m|\le r/2} + \int_{|m|\ge 2r} + \int_{\max(r-1,r/2)<|m|<2r} + \int_{r/2<|m|<r-1}\right) |\psi_k(m,r)|^2 \langle m\rangle\, dm \\ = & \ I + II + III + IV\ . \end{aligned}$$

The first two terms I and II can be estimated as before. For III,

$$\begin{aligned} III \ \le\ & C + \int_{\max(r-1,r/2)<|m|<2r} r^{-1}\langle |m|-r\rangle^{-2N}\langle m\rangle\, dm \\ \le\ & C + C\int_{\max(r-1,r/2)<|m|<2r} \langle |m|-r\rangle^{-2N} dm \le C. \end{aligned}$$

Now we turn to the estimate for IV,

$$\begin{aligned} IV \ \le\ & C + \int_{r/2<|m|<r-1} (r^{-1}\langle |m|-r\rangle^{-3} + r^{-1}\langle |m|-r\rangle^{-1}\min(k^2/d^2, d^2/k^2))\langle m\rangle\, dm \\ \le\ & C + C\int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-3} dm \\ & +C\int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \\ \le\ & C, \end{aligned}$$

where in the last inequality, we used the fact that

$$\int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \lesssim 1.$$

In fact, if $k^2\le r$, then

$$\begin{aligned} & \int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \\ \lesssim\ & \int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-1} k^2 d^{-2} dm \\ \le\ & C\int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-2} k^2 r^{-1} dm \\ \le\ & Ck^2/r \le C. \end{aligned}$$

Else, if $k^2 > r$, we have

$$\begin{aligned}
&\int_{r/2<|m|<r-1} \langle |m|-r\rangle^{-1} \min(k^2/d^2, d^2/k^2) dm \\
\leq\ & \int_{|m|<r-k^2/r} \langle |m|-r\rangle^{-1} k^2 d^{-2} dm \\
&+ \int_{\max(r/2, r-k^2/r)<|m|<r-1} \langle |m|-r\rangle^{-1} d^2 k^{-2} dm \\
\leq\ & C \int_{|m|<r-k^2/r} \langle |m|-r\rangle^{-2} k^2 r^{-1} dm \\
&+ C \int_{\max(r/2, r-k^2/r)<|m|<r-1} r k^{-2} dm \\
\leq\ & C \left\langle k^2/r\right\rangle^{-1} k^2/r + Crk^{-2} \min(k^2/r-1, r/2-1) \leq \tilde{C}.
\end{aligned}$$

This proves our key estimate (2.10).

Finally, we give the proof of Lemma 2.2, which will conclude the proof of (2.10).

Proof. If $r < m+1$, we have

$$m - r\cos\theta = r(1-\cos\theta) + m - r \geq m - r \geq -1.$$

Let $u = 1-\cos\theta$, so we get

$$\langle m - r\cos\theta\rangle \simeq 1 + r(1-\cos\theta) + (2+m-r) \simeq \langle m-r\rangle + \langle ru\rangle.$$

Since $\alpha \in \mathcal{S}$,

$$\begin{aligned}
|I_k(m,r)| &\leq \int_0^{\frac{3\pi}{4}} |\hat{\alpha}\big(m - r\cos\theta\big)|\, d\theta \\
&\leq C \int_0^{\frac{3\pi}{4}} \langle m - r\cos\theta\rangle^{-2N} d\theta \\
&\leq C \int_0^{\frac{3\pi}{4}} \langle m-r\rangle^{-N} \langle ru\rangle^{-N} d\theta \\
&\leq C \int_0^{1+1/\sqrt{2}} \langle m-r\rangle^{-N} \langle ru\rangle^{-N} \frac{du}{\sqrt{2u-u^2}} \\
&\leq C \int_0^{1+1/\sqrt{2}} \langle m-r\rangle^{-N} \langle ru\rangle^{-N} \frac{du}{\sqrt{u}} \\
&\leq Cr^{-1/2} \langle m-r\rangle^{-N},
\end{aligned}$$

which gives us (2.17).

Now we turn to the proof for the case $r \geq m+1$. We can imagine that the behavior is worst in the region that $m - r\cos\theta \sim 0$. To illustrate this, we introduce $\theta_0 \in (0, \frac{\pi}{2}]$ such that

$$r\cos\theta_0 = m,\ \sin\theta_0 = \frac{\sqrt{r^2-m^2}}{r} \equiv \frac{d}{r}. \tag{2.19}$$

Then the local behavior of the function $r\cos\theta - m$ near $\theta = \theta_0$ looks like

$$r\cos\theta - m \sim -d(\theta-\theta_0) + \mathcal{O}((\theta-\theta_0)^2),$$

since $r\cos\theta_0 - m = 0$ and $\frac{d}{d\theta}(r\cos\theta - m)|_{\theta=\theta_0} = -r\sin\theta_0 = -d$. Based on this information, we make the change of variable

$$\beta = d(\theta - \theta_0),\ \phi(\beta) = m - r\cos(\theta_0 + \beta/d)\ . \tag{2.20}$$

For the function ϕ, we can find that $\phi(0) = 0$, $\phi'(\beta) = \frac{r}{d}\sin\theta$. Moreover, we have the following

Lemma 2.3. *Let $\phi(\beta)$ be the function defined by* (2.20)*, and $\theta \in [\theta_1, \frac{3\pi}{4}]$ with $\theta_1 \in (0, \frac{\pi}{4})$ such that $\frac{r}{d}\sin\theta_1 = \frac{1}{2}$, then we have*

$$\frac{1}{2} \le \phi'(\beta) \le 1 + |\phi(\beta)| \simeq \langle \phi(\beta)\rangle\ . \tag{2.21}$$

In addition,

$$|\phi(\beta)| = |\phi(\beta) - \phi(0)| \ge \frac{1}{2}|\beta|.$$

Proof. We need only to give the proof of the inequality

$$\phi'(\beta) \le 1 + |\phi(\beta)|.$$

In fact, if $\beta = 0$ (i.e., $\theta = \theta_0$), we know the inequality is true with identity (see (2.19)). For $\beta \le 0$, we have $\phi(\beta) \le 0$ and the inequality amounts to $\phi'(\beta) \le 1 - \phi(\beta)$, which is equivalent to

$$\frac{r}{d}\sin\theta \le 1 + r\cos\theta - m,\ \theta \in [\theta_1, \theta_0].$$

Now we can see that this inequality is trivial by the monotonicity of the trigonometric functions

$$\frac{r}{d}\sin\theta \le \frac{r}{d}\sin\theta_0 = 1 = 1 + r\cos\theta_0 - m \le 1 + r\cos\theta - m.$$

If we consider instead the case $\beta \ge 0$, we know that it is equivalent to

$$\frac{r}{d}\sin\theta \le 1 + m - r\cos\theta,\ \theta \in [\theta_0, \frac{3\pi}{4}]. \tag{2.22}$$

Once again, by the monotonicity of the trigonometric functions, we need only to prove the inequality for $\theta \in [\theta_0, \frac{\pi}{2}]$. In the latter case, consider $F(\theta) = 1 + m - r\cos\theta - \frac{r}{d}\sin\theta$, we observe that (recall $r \ge m+1$)

$$F'(\theta) = r\sin\theta - \frac{r}{d}\cos\theta \ge r\sin\theta_0 - \frac{r}{d}\cos\theta_0 = d - \frac{m}{d} = \frac{d^2 - m}{d} \ge \frac{(m+1)^2 - m^2 - m}{d} \ge 0\ .$$

Recall $F(\theta_0) = 0$, we know that $F(\theta) \ge 0$ for $\theta \in [\theta_0, \frac{\pi}{2}]$ and so is (2.22). This completes the proof of the inequality (2.21). ∎

Now let us continue the proof of the estimate for I_k. We write

$$I_k(m,r) = J_k(m,r) + K_k(m,r) \tag{2.23}$$

with

$$\begin{aligned} J_k(m,r) &= \int_{\theta_1}^{3\pi/4} e^{-ik\theta}\hat{\alpha}\big(m - r\cos\theta\big)\ d\theta \\ &= \frac{e^{-ik\theta_0}}{d}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} e^{-i\frac{k}{d}\beta}\hat{\alpha}\big(\phi(\beta)\big)\ d\beta \equiv \frac{e^{-ik\theta_0}}{d}L_k. \end{aligned} \tag{2.24}$$

We first give the easier estimate for K_k. In fact, if $\theta \in [0, \theta_1]$,

$$r\cos\theta - m \geq r\cos\theta_1 - m = \frac{\sqrt{3r^2+m^2}}{2} - m \geq \frac{3(r^2-m^2)}{4\sqrt{3r^2+m^2}} \simeq r - m.$$

Note that $\theta_1 \sim \sin\theta_1 = \frac{d}{2r}$, this means that

$$|K_k(m,r)| \leq C\int_0^{\theta_1} (r-m)^{-N-1} d\theta \leq C\frac{d}{r}(r-m)^{-N-1} \leq Cr^{-1/2}(r-m)^{-N}. \tag{2.25}$$

Now we turn to the estimate for J_k in terms of L_k. We want to exploit the effect of oscillated factor $e^{-i\frac{k}{d}\beta}$, together with the support property of the function α. Recall that $i\frac{d}{k}\partial_\beta e^{-i\frac{k}{d}\beta} = e^{-i\frac{k}{d}\beta}$, we use integration by parts in β to get

$$\begin{aligned}
|L_k(m,r)| &= \left|\frac{d}{k}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} \partial_\beta(e^{-i\frac{k}{d}\beta})\hat{\alpha}\big(\phi(\beta)\big)\ d\beta\right| \\
&= \frac{d}{k}\left|e^{-i\frac{k}{d}\beta}\hat{\alpha}\big(\phi(\beta)\big)|_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} - \int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} e^{-i\frac{k}{d}\beta}\phi'(\beta)(\hat{\alpha})'\big(\phi(\beta)\big)\ d\beta\right| \\
&\leq C\frac{d}{k}\left(|\hat{\alpha}\big(m - r\cos\frac{3\pi}{4}\big)| + |\hat{\alpha}\big(m - r\cos\theta_1\big)| + \int_{\mathbb{R}} \langle\phi(\beta)\rangle^{1-N}\ d\beta\right) \\
&\leq C\frac{d}{k}\left(1 + \int_{\mathbb{R}} \langle\beta\rangle^{1-N}\ d\beta\right) \\
&\leq C\frac{d}{k}\ \text{, if } k \neq 0,
\end{aligned}$$

where we have used Lemma 2.3 in the first and second inequality.

To prove another inequality for $|L_k|$, we need only to exploit the support property of α. Since supp$\alpha \subset [\frac{1}{4}, 2]$, we can introduce $\tilde{\alpha}(\rho) = i\alpha(\rho)/\rho \in \mathcal{S}$ so that $\hat{\alpha} = (\hat{\tilde{\alpha}})'$ and

$$\hat{\alpha}(\phi(\beta)) = (\hat{\tilde{\alpha}})'(\phi(\beta)) = \frac{1}{\phi'(\beta)}\partial_\beta(\hat{\tilde{\alpha}}(\phi(\beta))).$$

Thus we have

$$
\begin{aligned}
|L_k(m,r)| &= \left|\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\partial_\beta(\hat{\tilde{\alpha}}(\phi(\beta)))\ d\beta\right| \\
&\le \left|\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\hat{\tilde{\alpha}}(\phi(\beta))\right)\right|_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} \\
&\quad - \int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} \partial_\beta\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\right)\hat{\tilde{\alpha}}(\phi(\beta))\ d\beta\Bigg| \\
&\le C\left(|\hat{\tilde{\alpha}}\big(m-r\cos\frac{3\pi}{4}\big)|+|\hat{\tilde{\alpha}}\big(m-r\cos\theta_1\big)|\right) \\
&\quad + \int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\partial_\beta\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\right)\hat{\tilde{\alpha}}(\phi(\beta))\ \right|d\beta \\
&\le C\langle r-m\rangle^{-N}+C\frac{k}{d}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\frac{1}{\phi'(\beta)}\hat{\tilde{\alpha}}(\phi(\beta))\ \right|d\beta \\
&\quad +C\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\frac{\phi''(\beta)}{(\phi'(\beta))^2}\hat{\tilde{\alpha}}(\phi(\beta))\ \right|d\beta \\
&\le C\langle r-m\rangle^{-1}+C\frac{k}{d}\ ,
\end{aligned}
$$

where we have used the fact that $r\cos\theta_1 - m \gtrsim r-m$, $\phi'(\beta)\ge\frac{1}{2}$ (for $\theta\in[\theta_1,\frac{3\pi}{4}]$) and $\phi''(\beta)=\frac{r}{d^2}\cos\theta=\mathcal{O}((r-m)^{-1})$. Combining with the previous inequality, we have proved

$$|L_k(m,r)|\le C\langle r-m\rangle^{-1}+C\min\left(\frac{k}{d},\frac{d}{k}\right)$$

and so is the inequality (2.18) by (2.23), (2.24) and (2.25). This completes the proof. ■

3. Almost global existence for cubic SLW

In this section, we prove Theorem 1.1, as an application of the endpoint estimate (1.6).

To begin, let us prove the fractional Leibniz rule in the Sobolev space with angular regularity.

Lemma 3.1. *Let $n=2$, $s\in(0,1)$, $b>\frac{1}{2}$ and $\psi\in\mathcal{S}(\mathbb{R}^2)$ be a radial function, then we have*

$$\|\psi * f\|_{L_r^\infty L_\theta^2}\lesssim\|\psi\|_{L^1}\|f\|_{L_r^\infty L_\theta^2}\ , \tag{3.1}$$

and the fractional Leibniz rule

$$\|fg\|_{H_\theta^{s,b}}\lesssim\|f\|_{L_{|x|}^\infty H_\theta^b}\|g\|_{H_\theta^{s,b}}+\|g\|_{L_{|x|}^\infty H_\theta^b}\|f\|_{H_\theta^{s,b}}\ . \tag{3.2}$$

Moreover, we have

$$\|fg\|_{H_\theta^{s,b}}\lesssim\|f\|_{L_{|x|}^\infty H_\theta^b\cap\dot H_\theta^{1,b}}\|g\|_{H_\theta^{s,b}}\ , \tag{3.3}$$

$$\|fg\|_{\dot H_\theta^{1,b}\cap L_{|x|}^\infty H_\theta^b}\lesssim\|f\|_{\dot H_\theta^{1,b}\cap L_{|x|}^\infty H_\theta^b}\|g\|_{\dot H_\theta^{1,b}\cap L_{|x|}^\infty H_\theta^b}\ . \tag{3.4}$$

Proof. At first, we give the proof for (3.1). Recall

$$(\psi * f)(x) = \int \psi(y) f(x-y) dy,$$

we set $x = (r\cos\omega, r\sin\omega)$, $y = (\lambda\cos\theta, \lambda\sin\theta)$, then $x - y = (\rho\cos\alpha, \rho\sin\alpha)$, with

$$\rho = \sqrt{r^2 + \lambda^2 - 2r\lambda\cos(\omega-\theta)},\ \alpha = \omega + \arcsin\left(\frac{\lambda}{\rho}\sin(\omega-\theta)\right).$$

Introducing a new variable $a = \omega - \theta \in [0, 2\pi]$, then $\rho = \rho(\lambda, r, a)$ and $\alpha = \alpha(\lambda, r, \omega, a) = \omega + h(\lambda, r, a)$ for some function h. Now, for fixed r,

$$\begin{aligned}
\|\psi * f\|_{L^2_\omega} &= \| \int_0^\infty \int_0^{2\pi} \psi(\lambda) f(x-y) \lambda d\lambda d\theta \|_{L^2_\omega} \\
&\le \|f(\rho\cos\alpha, \rho\sin\alpha)\|_{L^\infty_\lambda L^2_\omega L^1_\theta} \int_0^\infty |\psi(\lambda)| \lambda d\lambda \\
&\simeq \|\psi\|_{L^1} \|f(\rho\cos\alpha, \rho\sin\alpha)\|_{L^\infty_\lambda L^2_\omega L^1_a} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\alpha(\lambda, r, \omega, a), \rho(\lambda, r, a)\sin\alpha(\lambda, r, \omega, a))\|_{L^\infty_\lambda L^2_\omega L^2_a} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\omega, \rho(\lambda, r, a)\sin\omega)\|_{L^\infty_\lambda L^2_a L^2_\omega} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\omega, \rho(\lambda, r, a)\sin\omega)\|_{L^\infty_\lambda L^\infty_a L^2_\omega} \\
&\le \|\psi\|_{L^1} \|f(\rho\cos\omega, \rho\sin\omega)\|_{L^\infty_\rho L^2_\omega}\ ,
\end{aligned}$$

which proves (3.1). The estimate (3.1) tell us that the space $L^\infty_r L^2_\theta$ is stable under the frequency localization.

Based on (3.1), and the fact that H^b_θ is an algebra under multiplication when $b > 1/2$, we can easily apply Littlewood-Paley decomposition to prove the fraction Leibniz rule (3.2), (3.3), and (3.4). ■

Now we are ready to give the proof of Theorem 1.1, based on the endpoint estimate (1.6) and the fraction Leibniz rule (3.2).

At first, we prove the easier case when $P_\alpha(u)$ do not depend on u, for which the idea of the proof will be clear. After that, we will modify the proof to show that we can still handle the general case.

3.1. **The case with $P_\alpha(u) = C_\alpha$.** By (1.6) and energy estimate, for fixed $s > \frac{3}{2}$ and $b > \frac{1}{2}$, we have

$$\|e^{-itP} f\|_{L^2_T L^\infty_{|x|} H^b_\theta} \le C_0 (\ln(2+T))^{1/2} \|f\|_{H^{s-1,b}_\theta} \tag{3.5}$$

and

$$\|e^{-itP} f\|_{L^\infty_T H^{s-1,b}_\theta} \le C_0 \|f\|_{H^{s-1,b}_\theta} \tag{3.6}$$

with some constant $C_0 > 1$. Recall that we have the initial data $(u_0, u_1) \in H^{s,b}_\theta \cap H^{s-1,b}_\theta$ with

$$\|u_0\|_{H^{s,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta} = \epsilon \le \epsilon_0, \tag{3.7}$$

where ϵ_0 will be fixed later (see (3.9)).

Given the metric

$$d(u, v) = (\ln(2+T))^{-1/2} \|\partial_{t,x}(u-v)\|_{L^2_T L^\infty_{|x|} H^b_\theta} + \|\partial_{t,x}(u-v)\|_{L^\infty_T H^{s-1,b}_\theta}, \tag{3.8}$$

we define the complete domain with $T = \exp(c\epsilon^{-2})$ and $c \ll 1$ to be chosen later (see (3.9)),

$$X = \{u \in C_T H_\theta^{s,b} \cap C_T^1 H_\theta^{s-1,b} : u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ d(u,0) < \infty\}.$$

Then for any $u \in X$, we denote Πu to be the solution to the linear wave equation

$$\Box \Pi u = (\partial u)^\alpha$$

with initial data (u_0, u_1). Note that for $u \in X$, we have $(\partial u)^\alpha \in L_T^1 H_\theta^{s-1,b}$ by using the fraction Leibniz rule (3.2), and so $\Pi u \in C_T H_\theta^{s,b} \cap C_T^1 H_\theta^{s-1,b}$ is well defined. Thus, by energy estimates (3.6) and Strichartz estimates (3.5), we have

$$d(\Pi u, 0) \le C_1(\|u_0\|_{H_\theta^{s,b}} + \|u_1\|_{H_\theta^{s-1,b}}) + C_1\|(\partial u)^\alpha\|_{L_T^1 H_\theta^{s-1,b}}$$

for some $C_1 \ge C_0$. Based on this estimate, we define a complete domain $D_\epsilon \subset X$ so that the map Π will be a contraction map in D_ϵ (for ϵ_0 and c small enough),

$$D_\epsilon = \{u \in X;\ d(u,0) \le 2C_1\epsilon\}\ .$$

By using the fraction Leibniz rule (3.2) and noting that $T = \exp(c\epsilon^{-2}) > e > 2$, we have for some $C_2 \ge C_1$,

$$\begin{aligned} d(\Pi u, 0) &\le C_1(\|u_0\|_{H_\theta^{s,b}} + \|u_1\|_{H_\theta^{s-1,b}}) + C_1\|(\partial u)^\alpha\|_{L_T^1 H_\theta^{s-1,b}} \\ &\le C_1\epsilon + C_2\|\partial u\|^2_{L_T^2 L_{|x|}^\infty H_\theta^b}\|\partial u\|_{L_T^\infty H_\theta^{s-1,b}} \\ &\le C_1\epsilon + C_2 \ln(2+T) d(u,0)^3 \\ &\le C_1\epsilon + C_2 d(u,0)^3 + C_2 c\epsilon^{-2} d(u,0)^3\ . \end{aligned}$$

Moreover, for any $u, v \in X$, we have for some $C_3 \ge C_2$,

$$\begin{aligned} d(\Pi u, \Pi v) &\le C_1\|(\partial u)^\alpha - (\partial v)^\alpha\|_{L_T^1 H_\theta^{s-1,b}} \\ &\le C_3 \ln(2+T)(d(u,0)^2 + d(v,0)^2) d(u,v) \\ &\le (C_3 + C_3 c\epsilon^{-2})(d(u,0)^2 + d(v,0)^2) d(u,v). \end{aligned}$$

Now we fix the constant ϵ_0 and c such that we have

$$c\epsilon_0^{-2} \ge 1,\ 2C_2(2C_1\epsilon_0)^3 c\epsilon_0^{-2} \le C_1\epsilon_0,\ 2C_3 c\epsilon_0^{-2}(2 \times 2C_1\epsilon_0)^2 \le \frac{1}{2},$$

which can be satisfied if we set

$$c = \frac{1}{2^6 C_1^2 C_3},\ \epsilon_0 = \sqrt{c}\ . \tag{3.9}$$

Then we know that if $\epsilon \le \epsilon_0$, the map Π is a contraction map on the complete set D_ϵ, and the fixed point $u \in D_\epsilon$ of the map Π gives the required unique almost global solution.

3.2. **General case involving the unknown function.** In the general case, we need to give the estimate for u in $L_{t,x}^\infty$.

To begin, recall $u(t) = u(0) + \int_0^t \partial_t u(s) ds$, then we have the trivial bound

$$\|u(t)\|_{L^2} \le \|u(0)\|_{L^2} + t\|\partial_t u(s)\|_{L^\infty_{s\in[0,t]} L^2}\ . \tag{3.10}$$

This will give us the uniform bound in time of order 1, if we combine the bound of $\partial_{t,x} u$ in $L_t^\infty H^{s-1}$ $(s > 1)$. Note that we are in the situation that for $t \le \exp(c\epsilon^{-2})$, such a bound for u will not be admissible for our purpose.

Instead, we need to prove an improved estimates on the L^∞ norm on $u \in X$. We claim that we can in fact prove the uniform bound for u,

$$\|u\|_{L^\infty_{t\in[0,T_0]}L^\infty_{|x|}H^b_\theta} \le C \tag{3.11}$$

with C independent of ϵ, if $T_0 = \exp(\epsilon^{-2})$, $u \in D_\epsilon$ and $\|u(0,\cdot)\|_{H^{s,b}_\theta} \le \epsilon$. In fact, this estimate is a special case of the following lemma.

Lemma 3.2. *Let $n \ge 1$, $s > \frac{n}{2}$ and $s \ge 1$. Assume we have $u \in CH^s \cap C^1H^{s-1}([0,T_0]\times\mathbb{R}^n)$, then there is an universal constant C such that*

$$\|u(t)\|_{L^\infty} \le C\left(\frac{2}{n}\ \frac{1-\delta}{\delta}\right)^{\frac{1-\delta}{2}} (\|u(t)\|^{1-\delta}_{\dot H^{\frac{n}{2(1-\delta)}}}\|u(0)\|^\delta_{L^2} + t^\delta\|\partial_{t,x}u\|_{L^\infty_t H^{s-1}}) \tag{3.12}$$

for any $0<\delta\le 1-\frac{n}{2s}$ and $0\le t\le T_0$. In particular, if $n=2$, $T_0=\exp(\epsilon^{-2})$ and $\|u(0)\|_{L^2}+\|\partial_{t,x}u\|_{L^\infty_t H^{s-1}} \le C\epsilon$ with $\epsilon \ll 1$, then by choosing $\delta=\epsilon^2$, we have a uniform bound in ϵ

$$\|u\|_{L^\infty_{t,x}} \lesssim 1\ .$$

Moreover, if $n=2$, $T_0=\exp(\epsilon^{-2})$, $u(0)=0$ and $\|\partial_{t,x}u\|_{L^\infty_t H^{s-1}}<\infty$, then by choosing $\delta=\epsilon^2$, we have

$$\|u\|_{L^\infty_{t,x}} \le C\epsilon^{-1}\|\partial_{t,x}u\|_{L^\infty_t H^{s-1}}. \tag{3.13}$$

Proof. Recall that we have ($0<\delta<1$)

$$\|u\|_{L^\infty} \le C_\delta \|u\|^{1-\delta}_{\dot H^{\frac{n}{2(1-\delta)}}}\|u\|^\delta_{L^2}.$$

To prove the improved bound for u, we need to know the explicit dependence of C_δ with respect to $\delta\in(0,1)$,

$$\begin{aligned}
\|f\|_{L^\infty} &\le C\|\hat f\|_{L^1}\\
&\le C\||\xi|^{-\frac{n}{2}-a}|\xi|^{\frac{n}{2}+a}\hat f(\xi)\chi_{|\xi|\ge\lambda}\|_{L^1} + C\|\hat f(\xi)\chi_{|\xi|\le\lambda}\|_{L^1}\\
&\le C\||\xi|^{-\frac{n}{2}-a}\chi_{|\xi|\ge\lambda}\|_{L^2}\||\xi|^{\frac{n}{2}+a}\hat f(\xi)\|_{L^2} + C\lambda^{\frac{n}{2}}\|\hat f(\xi)\|_{L^2}\\
&\le \frac{C}{\sqrt a}\lambda^{-a}\|f\|_{\dot H^{\frac{n}{2}+a}} + C\lambda^{\frac{n}{2}}\|f\|_{L^2}\ .
\end{aligned}$$

By choosing

$$\lambda = \left(\frac{\|f\|_{\dot H^{\frac{n}{2}+a}}}{\sqrt a\|f\|_{L^2}}\right)^{\frac{1}{\frac{n}{2}+a}},$$

we conclude that

$$\|f\|_{L^\infty} \le Ca^{-\frac{n}{2n+4a}}\|f\|^{\frac{\frac{n}{2}}{\frac{n}{2}+a}}_{\dot H^{\frac{n}{2}+a}}\|f\|^{\frac{a}{\frac{n}{2}+a}}_{L^2}$$

i.e.

$$\|f\|_{L^\infty} \le C\left(\frac{2}{n}\ \frac{1-\delta}{\delta}\right)^{\frac{1-\delta}{2}}\|f\|^{1-\delta}_{\dot H^{\frac{n}{2(1-\delta)}}}\|f\|^\delta_{L^2} \tag{3.14}$$

From which we see that we can have the upper bound for C_δ

$$C_\delta \le C\left(\frac{2}{n}\ \frac{1-\delta}{\delta}\right)^{\frac{1-\delta}{2}}.$$

Now we can give the improved estimates for u. Observe that if we choose $\delta > 0$ small enough ($\delta \leq 1 - \frac{n}{2s}$ will be enough), we will have $\frac{n}{2(1-\delta)} \leq s$. Combining (3.14) with (3.10), we know that

$$\|u(t)\|_{L^\infty} \leq CC_\delta(\|u(t)\|^{1-\delta}_{\dot H^{\frac{n}{2(1-\delta)}}} \|u(0)\|^{\delta}_{L^2} + t^\delta \|\partial_{t,x} u\|_{L^\infty_t H^{s-1}}),$$

which gives us (3.12). ■

Now we are ready to prove the general case of Theorem 1.1. For any $u \in D_\epsilon$, we denote Πu to be the solution to the linear equation

$$\Box \Pi u = \sum_{|\alpha|=3} P_\alpha(u)(\partial u)^\alpha$$

with data (u_0, u_1). Recall that in the definition of the space X, we have assumed $T = T_\epsilon \equiv \exp(c\epsilon^{-2}) \leq \exp(\epsilon^{-2})$ since $c \leq 1$, and so that we have the bound (3.11)

$$\|u\|_{L^\infty_T L^\infty_{|x|} H^b_\theta} \lesssim 1 \ .$$

By (3.4) and $\|u\|_{L^\infty_T \dot H^{1,b}_\theta} \leq d(u,0) \leq 2C_1\epsilon \lesssim 1$, we know that

$$\|P_\alpha(u) - P_\alpha(0)\|_{L^\infty_T \dot H^{1,b}_\theta \cap L^\infty_T L^\infty_{|x|} H^b_\theta} \lesssim 1.$$

Thus, by (3.2) and (3.3), together with (3.5) and (3.6), we have for some $\tilde C_2 \geq C_2$

$$\begin{aligned}
d(\Pi u, 0) &\leq C_1(\|u_0\|_{H^{s,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta}) + C_1\|P_\alpha(u)(\partial u)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\
&\leq C_1\epsilon + C_1\|P_\alpha(0)(\partial u)^\alpha\|_{L^1_T H^{s-1,b}_\theta} + C_1\|(P_\alpha(u) - P_\alpha(0))(\partial u)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\
&\leq C_1\epsilon + 2C_2 c\epsilon^{-2} d(u,0)^3 \max_\alpha |P_\alpha(0)| + \tilde C_2 \|\partial u\|^2_{L^2_T L^\infty_{|x|} H^b_\theta} \|\partial u\|_{L^\infty_T H^{s-1,b}_\theta} \\
&\leq C_1\epsilon + 4\tilde C_2 c\epsilon^{-2} d(u,0)^3 \ .
\end{aligned}$$

Similarly, for any $u, v \in D_\epsilon$, we have $u(0) - v(0) = u_0 - u_0 = 0$ and so we can apply (3.13) for $u - v$ as follows

$$\|u - v\|_{L^\infty_{t,|x|} H^b_\theta} \leq \tilde C_4 \epsilon^{-1} \|\partial_{t,x}(u-v)\|_{L^\infty_t H^{s-1,b}_\theta}. \tag{3.15}$$

Recall again the fractional Leibniz rule (3.2)-(3.4), we have for some constant $\tilde C_3 \geq C_3$,

$$\begin{aligned}
d(\Pi u, \Pi v) &\leq C_1\|P_\alpha(u)(\partial u)^\alpha - P_\alpha(v)(\partial v)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\
&\leq C_1\|P_\alpha(u)((\partial u)^\alpha - (\partial v)^\alpha)\|_{L^1_T H^{s-1,b}_\theta} + C_1\|(P_\alpha(u) - P_\alpha(v))(\partial v)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\
&\leq \tilde C_3 \ln(2+T)(d(u,0)^2 + d(v,0)^2) d(u,v) + \tilde C_3 \ln(2+T) d(v,0)^3 \epsilon^{-1} d(u,v) \\
&\leq 4\tilde C_3 c\epsilon^{-2} \left(d(u,0)^2 + d(v,0)^2 + \epsilon^{-1} d(v,0)^3\right) d(u,v).
\end{aligned}$$

Now as in the case $P_\alpha = C_\alpha$, we can find some small positive constants c and ϵ_0, such that the previous estimates amount to

$$d(\Pi u, 0) \leq 2C_1\epsilon \text{ and } d(\Pi u, \Pi v) \leq \frac{1}{2} d(u,v),$$

for any $u, v \in D_\epsilon$ with $\epsilon \leq \epsilon_0$. Thus the map Π is a contraction map on the complete set D_ϵ, and the fixed point $u \in D_\epsilon$ of the map Π gives the required unique almost global solution.

4. Discussion

In this section, we consider the generalizations of Theorem 1.1 to the general problems. To begin, we consider the following Cauchy problem

$$\Box u = \sum_{|\alpha|=p} P_\alpha(u)(\partial u)^\alpha \equiv N(u) \tag{4.1}$$

on $\mathbb{R}\times\mathbb{R}^n$, with Cauchy data at time $t=0$

$$u(0,x) = u_0 \in H^s,\ \partial_t u(0,x) = u_1 \in H^{s-1}\ . \tag{4.2}$$

Before presenting our well posed results, we give a brief history of the study of these problems. In the case of classical C_0^∞ initial data with size of order ϵ, the global or almost global existence can be proved for $p \ge 1+\frac{2}{n-1}$ (global if $p > 1+\frac{2}{n-1}$ and almost global with lifespan $T_\epsilon \ge \exp(c\epsilon^{-(p-1)})$ if $p = 1+\frac{2}{n-1}$), see Sogge [10]. When $p = 1+\frac{2}{n-1}$, the lifespan T_ϵ is also sharp for the problem with nonlinearity $|\partial_t u|^p$ (Zhou [17]).

As in Theorem 1.1, our object here is to prove the corresponding results with low regularity. We will consider only the case $p \ge 3$, which can be easily handled by the linear Strichartz type estimates.

The scaling consideration tells us the critical Sobolev space for the equation is $\dot H^{s_c}$ with exponent

$$s_c = \frac{n+2}{2} - \frac{1}{p-1}\ , \tag{4.3}$$

which is then, heuristically, a lower bound for the range of permissible s such that the problem (1.1)-(1.2) is well-posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$. (See e.g. Theorem 2 in [3] for the ill posed result with $s < s_c$ and $N(u) = (\partial_t u)^p$.)

The local well posedness for the problem with low regularity has been extensively studied (see Ponce-Sideris [9], Tataru [16] and the authors [1]). It is proved that the problem is local well posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$ with $s > s_c$, if $p \ge 1+\frac{4}{n-1}$. When $p < 1+\frac{4}{n-1}$, there is another mechanism due to Lorentz invariance such that the problem is not well posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$ with $s = s_c + \epsilon$ for any $\epsilon > 0$ small enough (see e.g. Lindblad [6]). In this case ($2 \le p < 1+\frac{4}{n-1}$), the local well posedness is proved for $s > \frac{n+5}{4}$.

We will prove the corresponding results in the Sobolev spaces H^s with regularity $s \ge s_c$ (for $p > \max(1+\frac{4}{n-1}, 3)$) or $s > s_c$ (for $p = \max(1+\frac{4}{n-1}, 3)$ and $n \ne 3$). For the remained cases, we will need to use the Sobolev space with certain angular regularity $b > 0$ to establish the global existence. Now we are ready to state our existence results.

Theorem 4.1. *Let $n \ge 3$, $p \ge 3$ and $(n,p) \ne (3,3)$. Then if $p > 3$, the problem (4.1)-(4.2) is critically local well-posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$ such that $\partial u \in L^{p-1}_t L^\infty_x$, for any $s \ge s_c$. Moreover, if the data is sufficiently small in $H^s \times H^{s-1}$, then the solution is global. Furthermore, if $n \ge 4$, $p = 3$, we have the same result for any $s > s_c$.*

Remark 4. The reason we exclude $n = 2$ in this Theorem is that we do not have the Sobolev embedding

$$\dot H^s \cap \dot H^1 \subset L^\infty,$$

which is true only for $n \geq 3$. This is also the reason we need to use Lemma 3.2 to give the bound of u in the proof of Theorem 1.1. However, for the case we have $P_\alpha(u)$ are constants C_α, we can still have similar results for $n = 2$. Precisely, the problem is critical local well posed in $C_t H^s \cap C_t^1 H^{s-1}$ for $p > 5$ and $s \geq s_c$, and for $p = 5$ and $s > s_c$. It seems interesting to see whether the general problem (4.1) with $P_\alpha(u) \neq C_\alpha$ admits global solution in this setting.

The remained case is now $n = 2$ or $n = p = 3$. In the case $n = p = 3$, global result with small data in $H_\theta^{s_c,b}$ with any $b > 0$ has been proved in [7], by proving the angular Strichartz estimates

$$\|\partial u\|_{L_t^2 L_{|x|}^\infty L_\theta^r(\mathbb{R}\times\mathbb{R}^3)} \leq C_r(\|u_0\|_{H^2(\mathbb{R}^3)} + \|u_1\|_{H^1(\mathbb{R}^3)}),$$

for any $r < \infty$.

Now let us deal with the case $n = 2$ and $p \geq 4$. As we have seen in the Remark 4, we are able to prove global well posedness in $C_t H_x^s \cap C_t^1 H_x^{s-1}$ for small data when $P_\alpha(u) = C_\alpha$. However, in view of Lemma 3.2, it seems difficult to prove the corresponding result for the general case.

Instead, recall that in the study of the wave equations with nonlinearities involving u, we usually prove global existence by requiring that the initial data belongs to homogeneous Sobolev spaces $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ with $\gamma \in (0,1)$ (see e.g. [4] and Metcalfe-Sogge [8]). Inspired by these results, we can remedy this difficulty by requiring that the second initial data $g \in \dot{H}^{-\delta}$ in addition.

Now we can state our results for the case $n = 2$.

Theorem 4.2. *Let $n = 2$, $p \geq 5$ and $s \geq s_c$ (with $s > s_c$ when $p = 5$). Consider the equation (4.1)-(4.2) with fixed $\delta \in (0,1)$, there exist constants $\epsilon_0 > 0$ and $C > 0$ such that we have an unique global solution $u \in C_t H^s \cap C_t^1 H^{s-1}$ with*

$$\|\partial_{t,x} u\|_{L_t^\infty H^{s-1} \cap L_t^{p-1} L_x^\infty} + \epsilon \|\partial_{t,x} u\|_{L_t^\infty \dot{H}^{-\delta}} \leq C\epsilon,$$

whenever the initial data satisfies

$$\|\nabla u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}} + \epsilon\|u_0\|_{\dot{H}^{1-\delta}} + \epsilon\|u_1\|_{\dot{H}^{-\delta}} \leq \epsilon$$

with $\epsilon \leq \epsilon_0$.

In the remained case with $p = 4$, as in Theorem 1.1, we need to use Sobolev space with angular regularity as solution space.

Theorem 4.3. *Let $n = 2$, $p \geq 4$, $s \geq s_c$ and $b > 1/2$. Consider the equation (4.1)-(4.2) with fixed $\delta \in (0,1)$, there exist constants $\epsilon_0 > 0$ and $C > 0$ such that we have an unique global solution $u \in C_t H_\theta^{s,b} \cap C_t^1 H_\theta^{s-1,b}$ with*

$$\|\partial_{t,x} u\|_{L_t^\infty H_\theta^{s-1,b} \cap L_t^{p-1} L_{|x|}^\infty H_\theta^b} + \epsilon \|\partial_{t,x} u\|_{L_t^\infty \dot{H}_\theta^{-\delta,b}} \leq C\epsilon,$$

whenever the initial data satisfies

$$\|\nabla u_0\|_{H_\theta^{s-1,b}} + \|u_1\|_{H_\theta^{s-1,b}} + \epsilon\|u_0\|_{\dot{H}_\theta^{1-\delta,b}} + \epsilon\|u_1\|_{\dot{H}_\theta^{-\delta,b}} \leq \epsilon$$

with $\epsilon \leq \epsilon_0$.

Remark 5. For the case $p = 2$ with $P_\alpha(u) = C_\alpha$, there are some related works of Sterbenz [13] ($n \geq 6$), [15] ($n = 4$).

4.1. **Critical Local Well Posedness for $n \geq 3$.** Recall the classical Strichartz estimates (see [2], [5] for example), for the solution to the equation $\Box u = 0$ with initial data (u_0, u_1), we have

$$\|\partial_{t,x}u\|_{L_t^{p-1}L_x^\infty} + \|\partial_{t,x}u\|_{L_t^\infty H^{s-1}} \leq C(\|\nabla u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}), \tag{4.4}$$

if $p > \max(1 + \frac{4}{n-1}, 3)$ and $s \geq s_c$. Moreover, the same estimates are true if $p = \max(1 + \frac{4}{n-1}, 3)$, $s > s_c$ with $n \neq 3$.

Let

$$d_1^T(u,v) = \|\partial_{t,x}(u-v)\|_{L_t^{p-1}L_x^\infty([0,T]\times\mathbb{R}^n)},$$
$$d_2(u,v) = \|\partial_{t,x}(u-v)\|_{L_t^\infty H^{s-1}}.$$

We define the solution space to be the complete domain with $\epsilon \ll 1$ to be chosen later,

$$\begin{aligned} S = \quad & \{u \in C_t H^s \cap C_t^1 H^{s-1} : d_1^T(u,0) \leq 10C\epsilon, \\ & d_2(u,0) \leq 10C(\|\nabla u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}) := M\}, \end{aligned}$$

with metric $d(u,v) = d_1(u,v) + d_2(u,v)$.

For any $u \in S$, we denote Πu to be the solution to the linear equation

$$\Box \Pi u = P_\alpha(u)(\partial u)^\alpha$$

with data (u_0, u_1). Then, using (4.4), we have

$$d_1^T(\Pi 0, 0) + d_2(\Pi 0, 0) \leq C(\|\nabla u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}),$$

and hence $\lim_{T\to 0} d_1^T(\Pi 0, 0) = 0$. Thus given ϵ small enough, we can choose $T = T(\epsilon) > 0$ small enough (and $T = \infty$ when the initial data is small enough) such that

$$d_1^T(\Pi 0, 0) \leq C\epsilon.$$

This implies that $\Pi 0 \in S$.

Observe that we have

$$\|u\|_{L_{t,x}^\infty \cap L_t^\infty \dot{H}^{s-1,\frac{n}{s-1}}} \lesssim \|u\|_{L_t^\infty \dot{H}^1} + \|u\|_{L_t^\infty \dot{H}^s} \lesssim d_2(u,0)$$

if $n \geq 3$. If $u \in S$ and $\epsilon > 0$ is small enough, then by (4.4), Duhamel's principle and fractional Leibniz rule, we have

$$\begin{aligned} d_2(\Pi u, 0) \quad & \leq \quad C(\|\nabla u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}) + C\|P_\alpha(u)(\partial u)^\alpha\|_{L_t^1 H^{s-1}} \\ & \leq \quad M/10 + \tilde{C}(M)\|\partial u\|_{L_t^{p-1}L_x^\infty}^{p-1}\|\partial u\|_{L_t^\infty H^{s-1}} \\ & \leq \quad M/10 + \tilde{C}(M)(10C\epsilon)^{p-1}M \leq M, \end{aligned}$$

and similarly

$$\begin{aligned} d_1^T(\Pi u, 0) \quad & \leq \quad d_1^T(\Pi u - \Pi 0, 0) + d_1^T(\Pi 0, 0) \\ & \leq \quad C\|P_\alpha(u)(\partial u)^\alpha\|_{L_t^1 H^{s-1}} + C\epsilon \\ & \leq \quad C\epsilon + \tilde{C}(M)\|\partial u\|_{L_t^{p-1}L_x^\infty}^{p-1}\|\partial u\|_{L_t^\infty H^{s-1}} \\ & \leq \quad C\epsilon + \tilde{C}(M)(10C\epsilon)^{p-1}M \leq 10C\epsilon. \end{aligned}$$

This means that $\Pi u \in S$.

Moreover, for any $u, v \in S$, if $\epsilon \ll 1$ sufficiently small, we have

$$\begin{aligned} d(\Pi u, \Pi v) &\le C\|P_\alpha(u)(\partial u)^\alpha - P_\alpha(v)(\partial v)^\alpha\|_{L^1_T H^{s-1}} \\ &\le C\|P_\alpha(u)((\partial u)^\alpha - (\partial v)^\alpha)\|_{L^1_T H^{s-1}} + C\|(P_\alpha(u) - P_\alpha(v))(\partial v)^\alpha\|_{L^1_T H^{s-1}} \\ &\le \tilde{C}(M)(d_1^T(u,0)^{p-2} + d_1^T(v,0)^{p-2})(d(u,0) + d(v,0))d(u,v) \\ &\quad + \tilde{C}(M)(d_1^T(v,0))^{p-1})d_2(v,0)d_2(u,v) \\ &\le 4M\tilde{C}(M)(10C\epsilon)^{p-2}d(u,v) + M\tilde{C}(M)(10C\epsilon)^{p-1}d(u,v) \le \frac{1}{2}d(u,v). \end{aligned}$$

Thus we prove that the map Π is a contraction map on the complete set S, and then the fixed point $u \in S$ of the map Π gives the required unique solution.

4.2. **Global existence with small data for $n = 2$ and $p \ge 4$.** In this subsection, we prove Theorem 4.2 and Theorem 4.3. The proofs of both Theorems use essentially the same idea. So for simplicity, we give only the harder proof for Theorem 4.3, which involves the Sobolev space with angular regularity.

Recall that by Theorem 1.2, for the solution to the equation $\Box u = 0$ with initial data (u_0, u_1), we have

$$\|\partial_{t,x} u\|_{L_t^{p-1} L^\infty_{|x|} H^b_\theta \cap L^\infty_t H^{s-1,b}_\theta} \le C\|\nabla u_0\|_{H^{s-1,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta} := C\epsilon \tag{4.5}$$

for some constant $C > 1$. Note also that the initial data satisfy

$$\|u_0\|_{\dot{H}^{1-\delta,b}_\theta} + \|u_1\|_{\dot{H}^{-\delta,b}_\theta} \le 1.$$

Let

$$d_1(u,v) = \|\partial_{t,x}(u-v)\|_{L_t^{p-1} L^\infty_{|x|} H^b_\theta \cap L^\infty_t H^{s-1,b}_\theta} ,$$

$$d_2(u,v) = \epsilon\|\partial_{t,x}(u-v)\|_{L^\infty_t \dot{H}^{-\delta,b}_\theta} .$$

We define the solution space to be the complete domain with $\epsilon \ll 1$ to be chosen later,

$$S = \{u \in C_t H^{s,b}_\theta \cap C^1_t H^{s-1,b}_\theta : d_1(u,0) + d_2(u,0) \le 10C\epsilon\},$$

with metric $d(u,v) = d_1(u,v) + d_2(u,v)$.

As before, for any $u \in S$, we denote Πu to be the solution to the linear equation

$$\Box \Pi u = P_\alpha(u)(\partial u)^\alpha$$

with initial data (u_0, u_1). Then, using (4.5), we have

$$d_1(\Pi 0, 0) \le C\epsilon.$$

Moreover, by the energy estimates, we know that

$$d_2(\Pi 0, 0) \le \epsilon(\|u_0\|_{\dot{H}^{1-\delta,b}_\theta} + \|u_1\|_{\dot{H}^{-\delta,b}_\theta}) \le C\epsilon.$$

This implies that $\Pi 0 \in S$.

Observe that we have

$$\|u\|_{L^\infty_{t,|x|} L^2_\theta \cap L^\infty_t \dot{H}^1} \lesssim \|u\|_{L^\infty_{t,x} \cap L^\infty_t \dot{H}^1} \lesssim \|u\|_{L^\infty_t \dot{H}^{1-\delta}} + \|u\|_{L^\infty_t \dot{H}^s}$$

since $s > 1 = \frac{n}{2}$. Then for $u \in S$, we have

$$\|u\|_{L^\infty_{t,|x|} H^b_\theta \cap L^\infty_t \dot{H}^{1,b}_\theta} \lesssim 1 . \tag{4.6}$$

If $u \in S$ and $\epsilon > 0$ is small enough, then by (4.5), Duhamel's principle and fractional Leibniz rule (Lemma 3.1), we have

$$\begin{aligned} d_1(\Pi u, 0) &\leq C\epsilon + C\|P_\alpha(u)(\partial u)^\alpha\|_{L^1_t H^{s-1,b}_\theta} \\ &\leq C\epsilon + C_1\|\partial u\|^{p-1}_{L^{p-1}_t L^\infty_{|x|} H^b_\theta}\|\partial u\|_{L^\infty_t H^{s-1,b}_\theta} \\ &\leq C\epsilon + C_1 d_1(u,0)^p \leq 2C\epsilon\ . \end{aligned}$$

To give the estimate of d_2, we need to use the following generalized Strichartz estimates (see Remark 3 or Proposition 1.2 in [11])

$$\|e^{-itP}f\|_{L^q_t L^r_{|x|} L^2_\theta} \leq C_{q,r}\|f\|_{\dot{H}^{1-\frac{1}{q}-\frac{2}{r}}} \tag{4.7}$$

if $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$. We will only use the special case when

$$q = \frac{p-2}{\delta} \text{ and } 1 - \frac{1}{q} - \frac{2}{r} = \delta\ .$$

This choice of (q, r) satisfies the relation $\frac{p-1}{q'} = p - \frac{2}{r'}$. Note here that we have the admissible condition for (4.7) with this choice of (q, r) for small enough $\delta > 0$ if and only if $p > 3$.

Recall that by Sobolev embedding, the following embedding estimates are true

$$L^{2/(1+\delta)}_{|x|} L^2_\theta \subset L^{2/(1+\delta)} \subset \dot{H}^{-\delta} \text{ and } \dot{H}^{1-\delta} \subset L^{2/\delta}\ .$$

Thus by (4.6), (4.7), duality, Hölder inequality and the fact that H^b_θ is an algebra under multiplication when $b > 1/2$, we have

$$\begin{aligned} d_2(\Pi u, 0) &\leq d_2(\Pi u - \Pi 0, 0) + d_2(\Pi 0, 0) \\ &\leq \epsilon\|(P_\alpha(u) - P_\alpha(0))(\partial u)^\alpha\|_{L^1_t \dot{H}^{-\delta,b}_\theta} + \epsilon\|P_\alpha(0)(\partial u)^\alpha\|_{L^{q'}_t L^{r'}_{|x|} H^b_\theta} + C\epsilon \\ &\leq C\epsilon + \epsilon\|(P_\alpha(u) - P_\alpha(0))(\partial u)^\alpha\|_{L^1_t L^{2/(1+\delta)}_{|x|} H^b_\theta} + \epsilon|P_\alpha(0)|\|(\partial u)^\alpha\|_{L^{q'}_t L^{r'}_{|x|} H^b_\theta} \\ &\leq C\epsilon + \epsilon\|P_\alpha(u) - P_\alpha(0)\|_{L^\infty_t L^{2/\delta}_{|x|} H^b_\theta}\|(\partial u)^\alpha\|_{L^1_t L^2_{|x|} H^b_\theta} \\ &\quad + \epsilon|P_\alpha(0)|\|\partial u\|^{2/r'}_{L^\infty_t L^2_{|x|} H^b_\theta}\|\partial u\|^{p-2/r'}_{L^{p-1}_t L^\infty_{|x|} H^b_\theta} \\ &\leq C\epsilon + C_2(d_2(u,0) + \epsilon)d_1(u,0)^p \\ &\leq 2C\epsilon\ . \end{aligned}$$

This means that $\Pi u \in S$.

Moreover, for any $u, v \in S$, if $\epsilon \ll 1$ sufficiently small, we have

$$\begin{aligned} d_1(\Pi u, \Pi v) &\leq C\|P_\alpha(u)(\partial u)^\alpha - P_\alpha(v)(\partial v)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\ &\leq C\|P_\alpha(u)((\partial u)^\alpha - (\partial v)^\alpha)\|_{L^1_T H^{s-1}} + C\|(P_\alpha(u) - P_\alpha(v))(\partial v)^\alpha\|_{L^1_T H^{s-1}} \\ &\leq C_3(d_1(u,0)^{p-1} + d_1(v,0)^{p-1})d(u,v) \\ &\quad + C_3(d_1(v,0))^p\|P_\alpha(u) - P_\alpha(v)\|_{L^\infty_{t,|x|} H^b_\theta \cap L^\infty_t \dot{H}^{1,b}_\theta} \\ &\leq C_3(d_1(u,0)^{p-1} + d_1(v,0)^{p-1})d(u,v) + \tilde{C}_3\epsilon^{-1}(d_1(v,0))^p d(u,v) \\ &\leq (2^{p-1}C_3(10C\epsilon)^{p-1} + \tilde{C}_3\epsilon^{-1}(10C\epsilon)^p)d(u,v) \leq \frac{1}{2}d(u,v). \end{aligned}$$

As in the proof of $d_2(\Pi u, 0)$, we have

$$\begin{aligned} d_2(\Pi u, \Pi v) &\leq \epsilon \|(P_\alpha(u) - P_\alpha(0))((\partial u)^\alpha - (\partial v)^\alpha)\|_{L^1_t L^{2/(1+\delta)}_{|x|} H^b_\theta} \\ &\quad + \epsilon \|(P_\alpha(u) - P_\alpha(v))(\partial v)^\alpha\|_{L^1_t L^{2/(1+\delta)}_{|x|} H^b_\theta} \\ &\quad + \epsilon \|P_\alpha(0)((\partial u)^\alpha - (\partial v)^\alpha)\|_{L^{q'}_t L^{r'}_{|x|} H^b_\theta} \\ &\leq C_4 d_2(u, 0)(d_1(u, 0)^{p-1} + d_1(v, 0)^{p-1}) d_1(u, v) \\ &\quad + C_4 d_2(u, v) d_1(v, 0)^p \\ &\quad + C_4 \epsilon ((d_1(u, 0))^{p-1} + (d_1(v, 0))^{p-1}) d_1(u, v) \\ &\leq 10 C_4 (10 C \epsilon)^p d(u, v) \leq \frac{1}{2} d(u, v). \end{aligned}$$

Thus we prove that the map Π is a contraction map on the complete set S, and then the fixed point $u \in S$ of the map Π gives the required unique solution.

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